Bennett–Carl Inequalities for Symmetric Banach Sequence Spaces and Unitary Ideals

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Abstract

We prove an abstract interpolation theorem which interpolates the $(r,2)$-summing and $(s,2)$-mixing norm of a fixed operator in the image and range space. Combined with interpolation formulas for spaces of operators we obtain as an application the original Bennett–Carl inequalities for identities acting between Minkowski spaces ℓ_u as well as their analogues for Schatten classes $\mathcal{S}_u$. Furthermore, our techniques motivate a study of Bennett–Carl inequalities within a more general setting of symmetric Banach sequence spaces and unitary ideals.

In [Ben73] and [Car74] Bennett and Carl independently proved the following inequalities: For $1 \leq u \leq 2$ and $1 \leq u \leq v \leq \infty$ the identity operator $\mathrm{id} : \ell_u \hookrightarrow \ell_v$ is absolutely $(r,2)$-summing, i. e. there is a constant $c > 0$ such that for each set of finitely many $x_1, \ldots, x_n \in \ell_u$

$$\big(\sum_{k=1}^{n} \|x_k\|_{\ell_v}^r\big)^{1/r} \leq c \cdot \sup_{\|x'\|_{\ell_{u'}} \leq 1} \big(\sum_{k=1}^{n} |\langle x', x_k\rangle|^2\big)^{1/2},$$

if (and only if) $1/r \leq 1/u - \max(1/v, 1/2)$.

This result improved upon older ones of Littlewood and Orlicz, and is nowadays of extraordinary importance in the theory of eigenvalue distribution of power compact operators (see e. g. [Kön86] and [Pie87]). Later in [CD92] the "Bennett–Carl inequalities" were extended within the setting of so-called mixing operators (originally invented by Maurey [Mau74]): For $1 \leq u \leq 2$ and $1 \leq u \leq v \leq \infty$ every s-summing operator T defined on ℓ_v has a 2-summing restriction to ℓ_u if (and only if) $1/s \geq 1/2 - 1/u + \max(1/v, 1/2)$.

The crucial step in the proofs of Bennett and Carl is to establish the case $1 \leq u \leq v = 2$ which

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in terms of finite-dimensional spaces reads as follows: For $1 \leq u \leq v = 2$ and $2 \leq r \leq \infty$ such that $1/r = 1/u - 1/2$

$$\sup_n \pi_{r,2}(\ell_u^n \hookrightarrow \ell_2^n) < \infty, \tag{0.1}$$

where $\pi_{r,2}(\ell_u^n \hookrightarrow \ell_2^n)$ denotes the $(r,2)$-summing norm of the embedding $\ell_u^n \hookrightarrow \ell_2^n$.

Note that the formula $1/r = 1/u - 1/2$ occurs by a "naive interpolation" of the parameter r between the two well-known border cases

$$\sup_n \pi_{\infty,2}(\ell_2^n \hookrightarrow \ell_2^n) = \sup_n \|\ell_2^n \hookrightarrow \ell_2^n\| < \infty$$
$$\sup_n \pi_{2,2}(\ell_1^n \hookrightarrow \ell_2^n) = \sup_n \pi_2(\ell_1^n \hookrightarrow \ell_2^n) < \infty$$

(π_2 the 2-summing norm): For $1 \leq u \leq 2$ choose $0 \leq \theta \leq 1$ with $1/u = (1-\theta)/2 + \theta/1$, then $1/r = (1-\theta)/\infty + \theta/2 = 1/u - 1/2$.

Nevertheless the literature so far does not offer an approach to the Bennett–Carl inequalities within the framework of interpolation theory. We prove an abstract interpolation formula for the mixing norm of a fixed operator, and obtain as an application not only the original Bennett–Carl inequalities but also their "non-commutative" analogues for finite-dimensional Schatten classes. Moreover, we consider Bennett–Carl inequalities in a more general setting of symmetric Banach sequence spaces and unitary ideals, and apply these results to Orlicz and Lorentz sequence spaces.

For further extensions of the Bennett–Carl inequalities within the setting of Orlicz sequence spaces see a recent paper of Maligranda and Mastyło [MM99].

1 Preliminaries

If (a_n) and (b_n) are scalar sequences we write $a_n \prec b_n$ whenever there is some $c \geq 0$ such that $a_n \leq c \cdot b_n$ for all n, and $a_n \asymp b_n$ whenever $a_n \prec b_n$ and $b_n \prec a_n$. For $1 \leq p \leq \infty$ the number p' is defined by $1/p + 1/p' = 1$.

We shall use standard notation and notions from Banach space theory, as presented e. g. in [DJT95], [LT79] and [TJ89]. If E is a Banach space, then B_E is its (closed) unit ball and E' its dual. We consider complex Banach spaces only (but note that most of the consequences of our abstract interpolation results can be formulated also for real spaces). As usual $\mathcal{L}(E,F)$ denotes the Banach space of all (bounded and linear) operators from E into F endowed with the operator norm $\|\cdot\|$. With $\mathbf{T_2}(E)$ and $\mathbf{C_2}(E)$ we denote the (Gaussian) type 2 and cotype 2 constant of a Banach space E with this property, respectively, and with $\mathbf{M_{(u)}}(E)$ and $\mathbf{M^{(u)}}(E)$ the u-concavity and u-convexity constant of a Banach lattice E.

We call a Banach space $E \subset c_0$ (the space of all zero sequences) a symmetric Banach sequence space if the i-th standard unit vectors e_i form a symmetric basis, i. e. the e_i's form a Schauder basis such that $\|x\|_E = \|\sum_{i=1}^\infty \varepsilon_i x_{\pi(i)} e_i\|_E$ for each $x \in E$, each permutation π

of $\mathbb{N}$ and each choice of scalars ε_i with $|\varepsilon_i| = 1$. Moreover, denote for each n the subspace $\operatorname{span}\{e_i \,|\, 1 \leq i \leq n\}$ of E by E_n. Together with its natural order a symmetric Banach sequence space E forms a Banach lattice, and clearly its basis is 1-unconditional. The associated unitary ideal $\mathcal{S}_E$ is the Banach space of all compact operators $T \in \mathcal{L}(\ell_2, \ell_2)$ with singular numbers $(s_i(T))_i$ in E endowed with the norm $\|T\|_{\mathcal{S}_E} := \|(s_i(T))_i\|_E$; with $\mathcal{S}_E^n$ we denote $\mathcal{L}(\ell_2^n, \ell_2^n)$ together with the norm $\|T\|_{\mathcal{S}_E^n} := \|(s_i(T))_{i=1}^n\|_{E_n}$. For $E = \ell_u$ $(1 \leq u < \infty)$ one gets the well-known Schatten-u-class $\mathcal{S}_u$; for simplicity put $\mathcal{S}_\infty := \mathcal{L}(\ell_2, \ell_2)$.

For all information on Banach operator ideals, in particular on summing and mixing operators, see e. g. [DF93], [DJT95] and [Pie80]. An operator $T \in \mathcal{L}(E, F)$ is called absolutely (r,p)-summing $(1 \leq p \leq r \leq \infty)$ if there is a constant $\rho \geq 0$ such that

$$\Big(\sum_{i=1}^n \|Tx_i\|^r\Big)^{1/r} \leq \rho \cdot \sup\Big\{\Big(\sum_{i=1}^n |\langle x', x_i\rangle|^p\Big)^{1/p} \,\Big|\, x' \in B_{E'}\Big\}$$

for all finite sets of elements $x_1, \ldots, x_n \in E$ (with the obvious modifications for p or $r = \infty$). In this case, the infimum over all possible $\rho \geq 0$ is denoted by $\pi_{r,p}(T)$, and the Banach operator ideal of all absolutely (r,p)-summing operators by $(\Pi_{r,p}, \pi_{r,p})$; the special case $r = p$ gives the ideal (Π_p, π_p) of all absolutely p-summing operators.

An operator $T \in \mathcal{L}(E, F)$ is called (s,p)-mixing $(1 \leq p \leq s \leq \infty)$ whenever its composition with an arbitrary operator $S \in \Pi_s(F, Y)$ is absolutely p-summing; with the norm

$$\mu_{s,p}(T) := \sup\{\pi_p(ST) \,|\, \pi_s(S) \leq 1\}$$

the class $\mathcal{M}_{s,p}$ of all (s,p)-mixing operators forms again a Banach operator ideal. Obviously, $(\mathcal{M}_{p,p}, \mu_{p,p}) = (\mathcal{L}, \|\cdot\|)$ and $(\mathcal{M}_{\infty,p}, \mu_{\infty,p}) = (\Pi_p, \pi_p)$. Recall that due to [Mau74] (see also [DF93, 32.10–11]) summing and mixing operators are closely related:

$$(\mathcal{M}_{s,p}, \mu_{s,p}) \subset (\Pi_{r,p}, \pi_{r,p}) \qquad \text{for } 1/s + 1/r = 1/p,$$

and "conversely"

$$(\Pi_{r,p}, \pi_{r,p}) \subset (\mathcal{M}_{s_0,p}, \mu_{s_0,p}) \qquad \text{for } 1 \leq p \leq s_0 < s \leq \infty \text{ and } 1/s + 1/r = 1/p.$$

Moreover, it is known that each $(s,2)$-mixing operator on a cotype 2 space is even $(s,1)$-mixing (see again [Mau74] and [DF93, 32.2]).

For an operator $T \in \mathcal{L}(E, F)$ the n-th Weyl number $x_n(T)$ of T is defined by

$$x_n(T) := \sup\{a_n(TS) \,|\, S \in \mathcal{L}(\ell_2, E) \text{ with } \|S\| = 1\},$$

where $a_n(TS)$ denotes the n-th approximation number of TS. We will use the following important inequality of König in order to obtain lower estimates:

$$n^{1/r} \cdot x_n(T) \leq \pi_{r,2}(T), \qquad T \in \Pi_{r,2} \tag{1.1}$$

(for all details on s-numbers and this inequality see [Kön86, 2.a.3] or [Pie87]).

2 Complex Interpolation of Mixing Operators

The aim of this section is to prove a complex interpolation formula for the mixing norm of a fixed operator acting between two complex interpolation spaces.

For all information on complex interpolation we refer to [BL78]. A couple $[E_0, E_1]$ of Banach spaces is called an interpolation couple if there exists a topological Hausdorff vector space E in which E_0 and E_1 can be continuously embedded. We speak of a finite-dimensional interpolation couple $[E_0, E_1]$, if E_0 and E_1 are finite-dimensional Banach spaces with the same dimensions. For an interpolation couple $[E_0, E_1]$ of Banach spaces and $0 \le \theta \le 1$ we denote by $[E_0, E_1]_\theta = E_\theta$ the interpolation space obtained by the complex interpolation method of Calderón. Well-known examples of complex interpolation spaces are the Minkowski spaces $\ell_p^n(E)$ and Schatten classes $\mathcal{S}_p^n$: For $1 \le p_0, p_1 \le \infty$, $0 < \theta < 1$ and an interpolation couple $[E_0, E_1]$

$$[\ell_{p_0}^n(E_0), \ell_{p_1}^n(E_1)]_\theta = \ell_p^n(E_\theta) \qquad \text{and} \qquad [\mathcal{S}_{p_0}^n, \mathcal{S}_{p_1}^n]_\theta = \mathcal{S}_p^n$$

(isometrically), where $1/p = (1-\theta)/p_0 + \theta/p_1$ (for the complex interpolation formula for $\ell_p^n(E)$'s see [BL78, 5.1.2], whereas the formula for Schatten classes can be deduced from e. g. [PT68, Satz 8] and the complex reiteration theorem [BL78, 4.6.1]). For $0 \le \theta < 1$ a θ-Hilbert space is an interpolation space $[E_0, E_1]_\theta$ where E_1 is a Hilbert space (this notion goes back to Pisier); in particular, ℓ_p^n and $\mathcal{S}_p^n$ for $1 < p < \infty$ are θ-Hilbert spaces for $\theta = 1 - |1 - 2/p|$.

The following complex interpolation theorem for the mixing norm is our main abstract tool.

Theorem 1. *Let $2 \le s_0, s_1 \le \infty$, $0 \le \theta \le 1$ and s_θ given by $1/s_\theta = (1-\theta)/s_0 + \theta/s_1$. Then for two finite-dimensional interpolation couples $[E_0, E_1]$, $[F_0, F_1]$ and each $T \in \mathcal{L}(E_\theta, F_\theta)$*

$$\mu_{s_\theta,2}(T : E_\theta \to F_\theta) \le d_\theta[E_0, E_1] \cdot \mu_{s_0,2}(T : E_0 \to F_0)^{1-\theta} \cdot \mu_{s_1,2}(T : E_1 \to F_1)^\theta,$$

where

$$d_\theta[E_0, E_1] := \sup_m \|\mathcal{L}(\ell_2^m, E_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\|. \tag{2.1}$$

Proof. Consider for $\eta = 0, \theta, 1$ the bilinear mapping

$$\begin{array}{ccccccc} \Phi_\eta^{n,m} : & \ell_{s_\eta}^n(F'_\eta) & \times & \mathcal{L}(\ell_2^m, E_\eta) & \longrightarrow & \ell_2^m(\ell_{s_\eta}^n) \\ & (y'_1, \dots, y'_n) & \times & S & \longmapsto & ((\langle y'_k, TSe_j \rangle)_k)_j \end{array},$$

where (e_j) denotes the canonical basis in $\mathbb{C}^m$. By the discrete characterization of the mixing norm (see [Mau74] or [DF93, 32.4]) $\mu_{s_\eta}(T : E_\eta \to F_\eta)$ is the infimum over all $c \ge 0$ such that for all n, m, all $y'_1, \dots, y'_n \in F'_\eta$ and all $x_1, \dots, x_m \in E_\eta$

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle y'_k, Tx_j \rangle|^{s_\eta} \right)^{2/s_\eta} \right)^{1/2} \le c \cdot \left(\sum_{k=1}^n \|y'_k\|_{F'_\eta}^{s_\eta} \right)^{1/s_\eta} \cdot \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j \rangle|^2 \right)^{1/2}.$$

Since for each $S = \sum_{j=1}^m e_j \otimes x_j \in \mathcal{L}(\ell_2^m, E_\eta)$

$$\|S\| = \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j \rangle|^2 \right)^{1/2},$$

we obviously get that

$$\mu_{s_\eta,2}(T : E_\eta \to F_\eta) = \sup_{n,m} \|\Phi_\eta^{n,m}\|.$$

Now the proof follows by bilinear complex interpolation: For the interpolated bilinear mapping

$$[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta : [\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta \times [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta \longrightarrow [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta$$

by [BL78, 4.4.1]

$$\|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\| \le \|\Phi_0^{n,m}\|^{1-\theta} \cdot \|\Phi_1^{n,m}\|^\theta.$$

Since by the interpolation theorem for $\ell_p(E)$'s together with the duality theorem [BL78, 4.5.2]

$$[\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta = \ell_{s_\theta}^n(F'_\theta) \qquad \text{and} \qquad [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta = \ell_2^m(\ell_{s_\theta}^n)$$

(isometrically), we have

$$\|\Phi_\theta^{n,m}\| \le \|\mathcal{L}(\ell_2^m, E_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\| \cdot \|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\|.$$

Consequently

$$\begin{aligned}
\mu_{s_\theta,2}(T : E_\theta \to F_\theta) &= \sup_{n,m} \|\Phi_\theta^{n,m}\| \\
&\le \sup_{n,m} \{\|\mathcal{L}(\ell_2^m, E_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\| \cdot \|\Phi_0^{n,m}\|^{1-\theta} \cdot \|\Phi_1^{n,m}\|^\theta\} \\
&\le d_\theta[E_0, E_1] \cdot \mu_{s_0,2}(T : E_0 \to F_0)^{1-\theta} \cdot \mu_{s_1,2}(T : E_1 \to F_1)^\theta,
\end{aligned}$$

the desired result. □

In the same way an analogous result for the $(r,2)$-summing norm can be obtained.

Applications of theorem 1 come from "uniform estimates" for $d_\theta[E_0, E_1]$. Pisier proved in [Pi90] that

$$d_\theta[\ell_1, \ell_2] := \sup_n d_\theta[\ell_1^n, \ell_2^n] < \infty; \tag{2.2}$$

the proof is based on the Maurey factorization theorem which says that every operator $T : \ell_2 \to \ell_p$ $(1 \le p \le 2)$ factorizes through an appropriate diagonal operator $D_\lambda : \ell_2 \to \ell_p$, and spaces of diagonal operators clearly behave well under interpolation. The fact (2.2) can also be obtained as an application (case (c) of the following estimates) of a deep result of Kouba [Kou91] on the complex interpolation of injective tensor products of Banach spaces (see also [DM99])—as a consequence of Kouba's work we get for a finite-dimensional interpolation couple $[E_0, E_1]$ and $0 \le \theta \le 1$ the following estimates for $d_\theta[E_0, E_1]$:

(a) $d_\theta[E_0, E_1] = 1$, if $E_0 = E_1$.

(b) $d_\theta[E_0, E_1] \le \mathbf{T_2}(E_0')^{1-\theta} \cdot \mathbf{T_2}(E_1')^{\theta}$.

(c) $d_\theta[E_0, E_1] \le (2/\sqrt{\pi}) \cdot \mathbf{M_{(2)}}(E_0)^{1-\theta} \cdot \mathbf{M_{(2)}}(E_1)^{\theta}$, if the canonical bases in E_0 and E_1 are 1-unconditional and induce the lattice structures.

Note that the constants given in (b) and (c) are different from those in Kouba's work. They follow from a short analysis of his proofs in the finite-dimensional case: Since in our setting a Hilbert space is involved, Kouba's formulas (3.8) and (3.10) on p. 47–48 can both be changed into $\gamma_z(T) \le |||T|||_z$. Moreover, calculating the terms $W^E(z)$ defined in Kouba's lemma 3.2 and lemma 3.3 (use two spaces instead of a family of Banach spaces, see also [CCRSW82, p. 218]), one obtains

$$W^E(z) = \mathbf{T_2}(E_0)^{1-\theta(z)} \cdot \mathbf{T_2}(E_1)^{\theta(z)} \text{ and } W^E(z) = (2/\sqrt{\pi}) \cdot \mathbf{M^{(2)}}(E_0)^{1-\theta(z)} \cdot \mathbf{M^{(2)}}(E_1)^{\theta(z)}$$

(with $\theta(z)$ as in [CCRSW82, corollary 5.1]), respectively. This leads to the above estimates (note that E_0 and E_1 in Kouba's formulas, in our context have to be replaced by E_0' and E_1', and recall the well-known duality relation between 2-concavity and 2-convexity, see e. g. [LT79, 1.d.4]).

Later we also need a non-commutative version of Pisier's result (2.2). Using an extension of Kouba's formulas for the Haagerup tensor product of operator spaces due to [Pi96], Junge in [Jun96, 4.2.6] proved an analogue of (2.2) for Schatten classes:

$$d_\theta[\mathcal{S}_1, \mathcal{S}_2] := \sup_n d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] < \infty. \tag{2.3}$$

Finally we state a corollary on θ-Hilbert spaces which together with (2.2) and (2.3) is crucial for our purposes.

Corollary 1. *Let $0 \le \theta \le 1$, $E = [E_0, \ell_2^n]_\theta$ be a n-dimensional θ-Hilbert space and $2 \le s_\theta \le \infty$ given by $s_\theta = 2/\theta$. Then*

$$\mu_{s_\theta,2}(E \hookrightarrow \ell_2^n) \le d_\theta[E_0, \ell_2^n] \cdot \pi_2(E_0 \hookrightarrow \ell_2^n)^{1-\theta}.$$

3 Bennett–Carl Inequalities for Symmetric Banach Sequence Spaces

As announced the preceding interpolation theorem implies the Bennett–Carl result and its extension of Carl–Defant as an almost immediate consequence:

Corollary 2. *Let $1 \le u \le 2$ and $1 \le u \le v \le \infty$. Then for $2 \le s \le \infty$ such that $1/s = 1/2 - 1/u + \max(1/v, 1/2)$*

$$\sup_n \mu_{s,2}(\ell_u^n \hookrightarrow \ell_v^n) < \infty.$$

In particular, for $2 \le r \le \infty$ such that $1/r = 1/u - \max(1/v, 1/2)$

$$\sup_n \pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) < \infty.$$

Proof. Only the case $1 \leq u < v \leq 2$ has to be considered; the case $2 \leq v \leq \infty$ then easily follows by factorization through ℓ_2^n, and the case $u = v$ is trivial anyway. In what follows we use the complex interpolation formula for ℓ_p^n's without further mentioning.
i) Take first $v = 2$. It is well-known (see e. g. [Pie80, 22.4.8] or (3.4)) that

$$\pi_2(\ell_1^n \hookrightarrow \ell_2^n) = 1.$$

For $1 \leq u \leq 2$ choose $0 \leq \theta \leq 1$ such that $1/u = (1-\theta)/1 + \theta/2$. Then $s_\theta := 2/\theta = u'$, and by corollary 1 together with (2.2)

$$\mu_{u',2}(\ell_u^n \hookrightarrow \ell_2^n) \leq d_\theta[\ell_1, \ell_2] < \infty.$$

ii) Let $1 \leq u < v < 2$. Combining case i),

$$\mu_{u',2}(\ell_u^n \hookrightarrow \ell_2^n) \leq d_\theta[\ell_1, \ell_2],$$

and

$$\mu_{2,2}(\ell_u^n \hookrightarrow \ell_u^n) = \|\ell_u^n \hookrightarrow \ell_u^n\| = 1,$$

we have

$$\mu_{s_{\tilde\theta},2}(\ell_u^n \hookrightarrow \ell_v^n) \leq \sup_n d_{\tilde\theta}[\ell_u^n, \ell_u^n] \cdot d_\theta[\ell_1, \ell_2]^{1-\tilde\theta} < \infty,$$

with $\tilde\theta := (1/v - 1/2)/(1/u - 1/2)$ and $1/s_{\tilde\theta} := (1-\tilde\theta)/u' + \tilde\theta/2 = 1/2 - 1/u + 1/v = 1/s$. □

As in the original proofs of Bennett and Carl the crucial step in the preceding proof is to show that for the symmetric Banach sequence space $E = \ell_u$

$$\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty, \tag{3.1}$$

where $1 \leq u \leq 2$ and $1/r = 1/u - 1/2$. We now prove a result within the framework of symmetric Banach sequence spaces which shows that (3.1) is sharp in a very strong sense. Take an arbitrary 2-concave and u-convex Banach sequence space E—these geometric assumptions in particular imply that the continuous inclusions $\ell_u \subset E \subset \ell_2$ hold—which satisfies (3.1). The following result shows that there is only one such space:

Theorem 2. *Let $1 \leq u \leq 2$ and $1/r = 1/u - 1/2$. For each 2-concave and u-convex symmetric Banach sequence space E the following are equivalent:*

(1) $\sup_n \mu_{u',2}(E_n \hookrightarrow \ell_2^n) < \infty$.

(2) $\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty$.

(3) $E = \ell_u$.

Clearly we only have to deal with the implication (2) ⇒ (3); its proof is based on two lemmas. For the first one we invent the notion of "enough symmetries in the orthogonal group". Let $E = (\mathbb{C}^n, \|\cdot\|)$ be an n-dimensional Banach space. We say that E has *enough symmetries in* $\mathcal{O}(n)$ if there is a compact subgroup G in $\mathcal{O}(n)$ such that

$$\forall\, u \in \mathcal{L}(E)\, \forall\, g, g' \in G : \|u\| = \|gug'\| \tag{3.2}$$

and

$$\forall\, u \in \mathcal{L}(E) \text{ with } ug = gu \text{ for all } g \in G \,\exists\, c \in \mathbb{K} : u = c \cdot \mathrm{id}_E. \tag{3.3}$$

Basic examples of spaces with enough symmetries in the orthogonal group are the finite-dimensional spaces E_n and $\mathcal{S}_E^n$ associated to a symmetric Banach sequence space E. The following lemma extends the corresponding results in [CD97, p. 233, 236].

Lemma 1. *Let E_n and F_n have enough symmetries in $\mathcal{O}(n)$. Then*

$$\pi_2(E_n \hookrightarrow F_n) = n^{1/2} \cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|}, \tag{3.4}$$

and for $1 \leq k \leq n$

$$\left(\frac{n-k+1}{n}\right)^{1/2} \cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|} \leq x_k(E_n \hookrightarrow F_n) \leq \left(\frac{n}{k}\right)^{1/2} \cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|}. \tag{3.5}$$

Proof. (3.4): Trace duality allows to deduce the lower estimate from the upper one:

$$n \leq \pi_2(\ell_2^n \hookrightarrow F_n) \cdot \pi_2(F_n \hookrightarrow \ell_2^n) \leq \|\ell_2^n \hookrightarrow E_n\| \cdot \pi_2(E_n \hookrightarrow F_n) \cdot n^{1/2} \cdot \|\ell_2^n \hookrightarrow F_n\|^{-1}.$$

For the proof of the upper estimate it may be assumed without loss of generality that $F_n = \ell_2^n$ (factorize through ℓ_2^n). In this case it suffices to show that

$$\|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot B_{\ell_2^n}$$

is John's ellipsoid $D_{\max}$ of maximal volume in B_{E_n} (see e. g. [Pi89, 3.8] or [DJT95, 6.30]). By definition there is a linear bijection $u : \ell_2^n \to E_n$ such that $u(B_{\ell_2^n}) = D_{\max}$. In particular, $\|u\| = 1$ and $N(u^{-1}) = n$ (N denotes the nuclear norm, see e. g. [Pi89, 3.7] or [DJT95, 6.30]). On the other hand by a standard averaging argument there is a linear bijection $v : \ell_2^n \to E_n$ with $\|v\| = 1$, $N(v^{-1}) = n$ and $vg = gv$ for all $g \in G$, where G is a compact group in $\mathcal{O}(n)$ satisfying (3.2) and (3.3) (see [Pi89, 3.5] which also holds in the complex case). By property (3.3) of G and the fact that $\|v\| = 1$ we have $v = \|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot \mathrm{id}$. Then by Lewis' uniqueness theorem $v^{-1}u \in \mathcal{O}(n)$ ([Pi89, 3.7] or [DJT95, 6.25]). Altogether we finally obtain

$$\|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot B_{\ell_2^n} = v(B_{\ell_2^n}) = v[v^{-1}u(B_{\ell_2^n})] = u(B_{\ell_2^n}) = D_{\max}.$$

(3.5): Recall from (1.1) that $k^{1/2} \cdot x_k(T) \leq \pi_2(T)$ for every 2-summing operator T acting between two Banach spaces. Together with (3.4) this gives the second inequality. The first then follows from the basic properties of the Weyl numbers (see e. g. [Kön86]):

$$\begin{aligned} 1 &= x_n(\mathrm{id}_{\ell_2^n}) \\ &\leq x_k(\ell_2^n \hookrightarrow F_n) \cdot x_{n-k+1}(F_n \hookrightarrow \ell_2^n) \\ &\leq \|\ell_2^n \hookrightarrow E_n\| \cdot x_k(E_n \hookrightarrow F_n) \cdot \left(\frac{n}{n-k+1}\right)^{1/2} \cdot \|\ell_2^n \hookrightarrow F_n\|^{-1}. \end{aligned}$$

□

The following obvious examples will be useful later.

Corollary 3. *For $1 \leq u, v \leq \infty$*

$$\pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) = n \cdot \frac{\max(1, n^{1/v-1/2})}{\max(1, n^{1/u-1/2})} \tag{3.6}$$

and

$$x_{[n^2/2]}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp \frac{\max(1, n^{1/v-1/2})}{\max(1, n^{1/u-1/2})}. \tag{3.7}$$

The preceding lemma turns out to be of special interest in combination with a result due to Szarek and Tomczak-Jaegermann [STJ80, proposition 2.2] which states that for each 2-concave symmetric Banach sequence space E

$$\|\ell_2^n \hookrightarrow E_n\| \asymp n^{-1/2} \cdot \|\sum\nolimits_1^n e_i\|_{E_n}. \tag{3.8}$$

The second lemma which we need for the proof of theorem 2, is based on (3.8) and an important result about the interpolation of Banach lattices due to Pisier [Pi79].

Lemma 2. *For $1 \leq u \leq 2$ let E be a u-convex and u'-concave symmetric Banach sequence space. Then*

$$\|E_n \hookrightarrow \ell_u^n\| \asymp \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}}. \tag{3.9}$$

In particular, if E is even 2-concave, then

$$\|E_n \hookrightarrow \ell_u^n\| \asymp \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}} \asymp \frac{n^{1/u-1/2}}{\|\ell_2^n \hookrightarrow E_n\|}. \tag{3.10}$$

Proof. (3.10) follows directly from (3.9) and (3.8), and clearly $n^{1/u} \leq \|E_n \hookrightarrow \ell_u^n\| \cdot \|\sum_1^n e_i\|_{E_n}$. For the upper estimate in (3.9) we only have to consider $1 < u < 2$: The case $u = 1$ is stated below in (3.11), and a 2-convex and 2-concave symmetric Banach sequence space necessarily equals ℓ_2 with equivalent norms. Without loss of generality we may assume $\mathbf{M}^{(\mathbf{u})}(E) = \mathbf{M}_{(\mathbf{u'})}(E) = 1$ (see [LT79, 1.d.8]). Then by [Pi79, theorem 2.2] there exists a symmetric Banach sequence space E_0 such that $E = [E_0, \ell_2]_\theta$ with $\theta = 2/u'$; moreover, we have $E_n = [E_0^n, \ell_2^n]_\theta$ with equal norms. The conclusion now follows by interpolation: It can be shown easily that

$$\|E_0^n \hookrightarrow \ell_1^n\| \leq \frac{n}{\|\sum_1^n e_i\|_{E_0^n}} \tag{3.11}$$

(see e. g. [STJ80, proposition 2.5]), hence

$$\|E_n \hookrightarrow \ell_u^n\| \leq \|E_0^n \hookrightarrow \ell_1^n\|^{1-\theta} \cdot \|\ell_2^n \hookrightarrow \ell_2^n\|^\theta \leq \frac{n^{1-\theta}}{\|\sum_1^n e_i\|_{E_0^n}^{1-\theta}}.$$

Since $E_n = [E_0^n, \ell_2^n]_\theta$ is of J-type θ (i. e. $\|x\|_{E_n} \leq \|x\|_{E_0^n}^{1-\theta} \cdot \|x\|_{\ell_2^n}^\theta$ for all $x \in E_n$), we have

$$\|\sum\nolimits_1^n e_i\|_{E_n} \leq \|\sum\nolimits_1^n e_i\|_{E_0^n}^{1-\theta} \cdot n^{\theta/2},$$

and consequently

$$\|E_n \hookrightarrow \ell_u^n\| \le \frac{n^{1-\theta/2}}{\|\sum_1^n e_i\|_{E_n}} = \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}}.$$

□

Proof of the implication (2) ⇒ (3) in theorem 2: Assume that $\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty$. By (1.1), (3.5) and (3.10)

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \ge [n/2]^{1/r} \cdot x_{[n/2]}(E_n \hookrightarrow \ell_2^n) \succ \frac{n^{1/r}}{\|\ell_2^n \hookrightarrow E_n\|} \asymp \|E_n \hookrightarrow \ell_u^n\|, \tag{3.12}$$

which by assumption shows that $\sup_n \|E_n \hookrightarrow \ell_u^n\| < \infty$. This clearly gives the claim. □

Note that (3.12) does not depend on the special choice of r.

If E is a 2-concave and u-convex ($1 \le u \le 2$) symmetric Banach sequence space different from ℓ_u (i. e. the inclusion $\ell_u \subset E$ is strict), then by theorem 2 for $1/r = 1/u - 1/2$

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \nearrow \infty.$$

The following result gives the precise asymptotic order of the sequence $(\pi_{r,2}(E_n \hookrightarrow \ell_2^n))_n$:

Corollary 4. *For $1 \le u \le 2$ let E be a 2-concave and u-convex symmetric Banach sequence space. Then for $2 \le r, s \le \infty$ such that $1/r = 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(E_n \hookrightarrow \ell_2^n) \asymp \frac{n^{1/r+1/2}}{\|\sum_1^n e_i\|_{E_n}}.$$

Proof. The lower estimate has already been shown in (3.12), and the upper estimate simply follows by factorization through ℓ_u^n, the Bennett–Carl inequalities and (3.10). □

Remark 1. (a) Since a u-convex Banach lattice is p-convex for all $1 \le p \le u$ (see [LT79, 1.d.5]), the formula in the preceding theorem even holds for all $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/2$.

(b) For $1 \le u \le 2$ let E be a 2-concave and u-convex symmetric Banach sequence space, F an arbitrary symmetric Banach sequence space, and let $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/2$. Then—by factorization through ℓ_2^n for the upper estimate and (3.12) for the lower one—the following formula holds:

$$\pi_{r,2}(E_n \hookrightarrow F_n) \asymp n^{1/r} \cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|};$$

in particular, if F is a 2-concave, then

$$\pi_{r,2}(E_n \hookrightarrow F_n) \asymp n^{1/r} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}}.$$

Note that these results can be considered as extensions of (3.4).

(c) For the special case $F = \ell_v$ ($1 \le u \le v \le 2$) the formulas in (b) even hold for all $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/v$; simply repeat the proof of corollary 4 for $1/r = 1/u - 1/v$ and use the argument from remark (a).

4 Bennett–Carl Inequalities for Unitary Ideals

We now use Junge's counterpart (2.3) of (2.2) and our interpolation theorem 1 in order to show a "non-commutative" analogue. Note first that for all $1 \le u, v \le \infty$ and $2 \le r \le \infty$

$$n^{1/r} \le \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n), \tag{4.1}$$

and hence also for $2 \le s \le \infty$

$$n^{1/2-1/s} \le \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n);$$

this is a consequence of the trivial estimate $\pi_{r,2}(\ell_2^n \hookrightarrow \ell_2^n) \ge n^{1/r}$ (insert e_k's) and the fact that ℓ_2^n is 1-complemented in each $\mathcal{S}_u^n$ (assign to each $x \in \ell_2^n$ the matrix $x \otimes e_1 \in \mathcal{S}_u^n$). For u, v considered in corollary 2 this lower bound is optimal:

Corollary 5. *Let $1 \le u \le 2$ and $1 \le u \le v \le \infty$. Then for $2 \le s \le \infty$ such that $1/s = 1/2 - 1/u + \max(1/v, 1/2)$*

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/2-1/s}.$$

In particular, for $2 \le r \le \infty$ and $1/r = 1/u - \max(1/v, 1/2)$

$$\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/r}.$$

Proof. The proof of the upper bound is analogous to that of corollary 2: Of course the complex interpolation formula for $\mathcal{S}_p^n$'s is needed instead of that for ℓ_p^n's, and in i) use $\pi_2(\mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = n^{1/2}$ (see (3.6)) and Junge's result (2.3) in order to obtain

$$\mu_{u',2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \le d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] \cdot n^{(1-\theta)/2} \le d_\theta[\mathcal{S}_1, \mathcal{S}_2] \cdot n^{1/u-1/2},$$

where $\theta = 2/u'$. Then in ii) one arrives at

$$\mu_{s_{\tilde\theta},2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \prec n^{(1-\tilde\theta)(1/u-1/2)} = n^{1/u-1/v},$$

with $\tilde\theta := (1/v - 1/2)/(1/u - 1/2)$ and $1/s_{\tilde\theta} = (1-\tilde\theta)/u' + \theta/2 = 1/2 - 1/u + 1/v$=1/s. $\square$

Exploiting the ideas of the preceding section one easily obtains the asymptotic order of the $(r, 2)$-summing and the $(s, 2)$-mixing norm of identities between finite-dimensional unitary ideals $\mathcal{S}_E^n$ and $\mathcal{S}_2^n$:

Corollary 6. *For $1 \le u \le 2$ let E be a 2-concave and u-convex symmetric Banach sequence space. Then for all $2 \le r, s \le \infty$ such that $1/r \ge 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}}.$$

Proof. Recall the simple fact that for all symmetric Banach sequence spaces E and F

$$\|\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n\| = \|E_n \hookrightarrow F_n\|, \tag{4.2}$$

and by the same reasoning as in remark 1 (a) it is enough to deal with the case $1/r = 1/u - 1/2$. Then factorization through $\mathcal{S}_u^n$ and (3.10) give

$$\mu_{u',2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \prec \|\mathcal{S}_E^n \hookrightarrow \mathcal{S}_u^n\| \cdot n^{1/u-1/2} \asymp \frac{n^{2/u-1/2}}{\|\sum_1^n e_i\|_{E_n}} = \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}},$$

and in order to obtain the lower estimate apply again (3.5) together with (1.1) and the second asymptotic in (3.10):

$$\pi_{r,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \geq [n^2/2]^{1/r} \cdot x_{[n^2/2]}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \succ \frac{n^{2/r}}{\|\ell_2^n \hookrightarrow E_n\|} \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}}.$$

□

5 Applications

Weyl Numbers

The results of the preceding sections can be used to improve the estimates for Weyl numbers of identities on symmetric Banach sequence spaces and unitary ideals in (3.5): The exponent $1/2$ in each of the two inequalities there can be replaced by $1/u - 1/2$ whenever u-convexity and 2-concavity assumptions are made.

Corollary 7. *For $1 \leq u, v \leq 2$ let E and F be 2-concave symmetric Banach sequence spaces where E is u-convex and F is v-convex. Then there exist constants $C_u, C_v > 0$ such that for all $1 \leq k \leq n$*

$$C_v^{-1} \cdot \left(\frac{n-k+1}{n}\right)^{1/v-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}} \leq x_k(E_n \hookrightarrow F_n) \leq C_u \cdot \left(\frac{n}{k}\right)^{1/u-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}},$$

and all $1 \leq k \leq n^2$

$$C_v^{-1} \cdot \left(\frac{n^2-k+1}{n^2}\right)^{1/v-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}} \leq x_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \leq C_u \cdot \left(\frac{n^2}{k}\right)^{1/u-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}}.$$

Proof. The upper estimates follow by using the inequality (1.1) and the results from the preceding two sections, and the lower estimates then are immediate consequences of the upper ones—simply repeat the proof of (3.5) with a different exponent. □

Recall that for the embedding $\ell_u^n \hookrightarrow \ell_2^n$, $1 \leq u \leq 2$ by [CD92, 2.3.3] even the following equality is known: $x_k(\ell_u^n \hookrightarrow \ell_2^n) = k^{1/2-1/u}$, $1 \leq k \leq n$. The second estimate in corollary 7 implies that for $1 < u < 2$

$$x_k(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \leq C_u \cdot \left(\frac{n}{k}\right)^{1/u-1/2},$$

hence by [CD92, 2.3.2]

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) \geq C_u^{-1} \cdot \left(\frac{n^2-k+1}{n}\right)^{1/u-1/2}.$$

This disproves the conjecture

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) \asymp \max\left(1, \left(\frac{n^2-k+1}{n^2}\right)^{1/2} \cdot n^{1/u-1/2}\right)$$

from [CD97, p. 249] (put $k := [n^2 - n^\alpha + 1]$, $1 < \alpha < 2$).

Identities on Orlicz and Lorentz Sequence Spaces

In the following we apply our results, in particular the corollaries 4 and 6, to two natural examples of symmetric sequence spaces: Orlicz and Lorentz sequence spaces (for their definition and basic properties we refer to [LT77]). We only treat the case where the range space of the embedding is the finite-dimensional Hilbert space and leave the formulation for other spaces and the corollaries for Weyl numbers to the reader.

Let us start with Orlicz sequence spaces ℓ_M.

Corollary 8. *Let $1 < u < 2$ and M be a strictly increasing Orlicz function which satisfies the Δ_2-condition at zero. Assume that there exists $K > 0$ such that for all $s, t \in (0, 1]$*

$$K^{-1} \cdot s^2 \leq M(st)/M(t) \leq K \cdot s^u. \tag{5.1}$$

Then for $2 < r, s < \infty$ such that $1/r > 1/u - 1/2$ and $1/s = 1/2 - 1/r$

$$\pi_{r,2}(\ell_M^n \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(\ell_M^n \hookrightarrow \ell_2^n) \asymp \frac{n^{1/r+1/2}}{\|\sum_1^n e_i\|_{\ell_M^n}} \asymp n^{1/r+1/2} \cdot M^{-1}(1/n)$$

and

$$\pi_{r,2}(\mathcal{S}_{\ell_M}^n \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}_{\ell_M}^n \hookrightarrow \mathcal{S}_2^n) \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{\ell_M^n}} \asymp n^{2/r+1/2} \cdot M^{-1}(1/n).$$

Note that (5.1) together with the Δ_2-condition assures that ℓ_M is 2-concave and p-convex for all $1 \leq p < u$ (see [LT79, 2.b.5]).

Now we state an analogue for Lorentz sequence spaces $d(w, u)$.

Corollary 9. *Let $1 < u < 2$ and w be such that $n \cdot w_n^q \asymp \sum_{i=1}^n w_i^q$, where $q = 2/(2-u)$. Then for $2 < r, s < \infty$ such that $1/r \geq 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(d_n(w,u) \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(d_n(w,u) \hookrightarrow \ell_2^n) \asymp n^{1/r+1/2-1/u} \cdot w_n^{-1/u}.$$

and

$$\pi_{r,2}(\mathcal{S}_{d(w,u)}^n \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}_{d(w,u)}^n \hookrightarrow \mathcal{S}_2^n) \asymp n^{2/r+1/2-1/u} \cdot w_n^{-1/u}.$$

Recall that the space $d(w, u)$ is u-convex, and if $1 \leq u < 2$, it is 2-concave if and only if w satisfies the condition in the assumption of the corollary (see [Rei81, p. 245–247]).

Limit Orders

Finally, we consider the asymptotic order of the sequences $(\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n))_n$ for arbitrary $2 \le r \le \infty$, $1 \le u, v \le \infty$. Define the limit orders

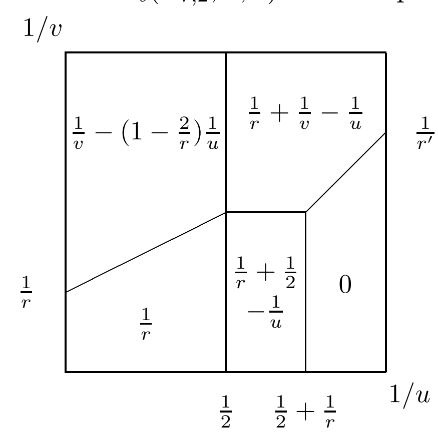
$$\lambda_\ell(\Pi_{r,2}, u, v) := \inf\{\lambda > 0 \,|\, \exists\, \rho > 0 \,\forall\, n : \pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) \le \rho \cdot n^\lambda\}$$

and

$$\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) := \inf\{\lambda > 0 \,|\, \exists\, \rho > 0 \,\forall\, n : \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le \rho \cdot n^\lambda\}.$$

Here we only handle the limit order of summing operators since—using the fact that $\Pi_{r,2}$ and $\mathcal{M}_{s,2}$ for $1/s + 1/r = 1/2$ are almost equal—one can easily see that $\lambda_\ell(\Pi_{r,2}, u, v) = \lambda_\ell(\mathcal{M}_{r,2}, u, v)$ and $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = \lambda_{\mathcal{S}}(\mathcal{M}_{r,2}, u, v)$ (with the obvious definition for the right sides of these equalities; see [Pie80, 22.3.7]).

The calculation of the limit order $\lambda_\ell(\Pi_{r,2}, u, v)$ was completed in [CMP78]:



Moreover, the proof in [CMP78] shows that the limit order is attained: $\pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) \asymp n^{\lambda_\ell(\Pi_{r,2},u,v)}$. In view of the results of section 4 the following conjecture seems to be natural:

Conjecture: $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = 1/r + \lambda_\ell(\Pi_{r,2}, u, v)$.

For the border cases $r = 2$ (the 2-summing norm) and $r = \infty$ (the operator norm) this conjecture by (3.4) and (3.6) is true. In the following corollary we confirm the upper estimates of this conjecture for all u, v and the lower ones for all u, v except those in the upper left corner of the picture.

Corollary 10. *Let $1 \le u, v \le$ and $2 < r < \infty$.*

(a) $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = 1/r + \lambda_\ell(\Pi_{r,2}, u, v)$ *for* $1 \le u \le 2$.

(b) $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) \le 1/r + \lambda_\ell(\Pi_{r,2}, u, v)$ *for* $2 \le u \le \infty$, *with equality whenever* $1/v \le 1/r + (1 - 2/r)(1/u)$.

Proof. Let $1/s := 1/2 - 1/r$. The upper estimates for the case $1 \le u \le 2$ follow from corollary 5: Consider for $u_0 := (1/2 - 1/r)^{-1}$ the following alternative: (i) $1/u \le 1/u_0$ or (ii)

$1/u > 1/u_0$. Then the conclusion in case (i) is a consequence of corollary 5 and the following factorization:

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{u_0}^n\| \cdot \mu_{s,2}(\mathcal{S}_{u_0}^n \hookrightarrow \mathcal{S}_2^n) \cdot \|\mathcal{S}_2^n \hookrightarrow \mathcal{S}_v^n\| \prec n^{2/r+1/2-1/u+\max(0,1/v-1/2)},$$

and for (ii) look with $v_0 := (1/u - 1/r)^{-1} \leq 2$ at

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_0}^n) \cdot \|\mathcal{S}_{v_0}^n \hookrightarrow \mathcal{S}_v^n\| \prec n^{1/r+\max(0,1/r+1/v-1/u)}.$$

Now let $2 \leq u \leq \infty$. Although this part is very close to the calculations made in [CMP78, Lemma 6], we give a short sketch of the proof for the convenience of the reader. By (3.6) and theorem 1 (with no interpolation in the range or the image),

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \leq \pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{2/r} \cdot \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n\|^{1-2/r} = n^{1/r+1/2-(1-2/r)(1/u)},$$

hence, by factorization, for $1 \leq v \leq 2$

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq n^{1/r+1/v-(1-2/r)(1/u)}.$$

Furthermore, for $1/v_1 := 1/r + (1-2/r)(1/u)$

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_1}^n) \leq \pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{2/r} \cdot \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_u^n\|^{1-2/r} = n^{2/r},$$

hence

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq n^{2/r}$$

for all $v_1 \leq v \leq \infty$. Finally, for all $2 < v < v_1$ and $0 < \theta < 1$ such that $1/v = (1-\theta)/v_1 + \theta/2$

$$\begin{aligned}\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) &\leq \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_1}^n)^{1-\theta} \cdot \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{\theta} \\ &\leq n^{1/r+(1-\theta)/r+\theta(1/2-(1-2/r)(1/u))} = n^{1/r+1/v-(1-2/r)(1/u)}.\end{aligned}$$

Looking at the picture for $\lambda_\ell(\Pi_{r,2}, u, v)$ one can see that these are the desired results. For the lower estimates recall (1.1):

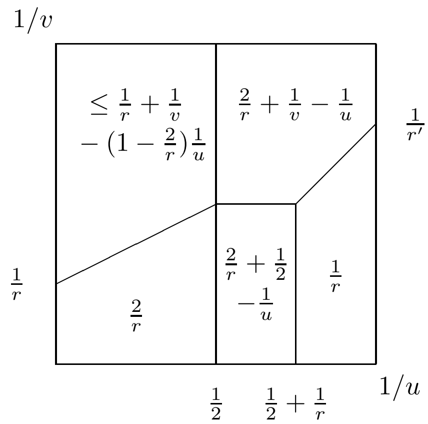
$$[n^2/2]^{1/r} \cdot x_{[n^2/2]}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n),$$

hence (3.7) implies

$$\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \succ \begin{cases} n^{2/r+1/v-1/u} & \text{if } 1 \leq u, v \leq 2, \\ n^{2/r+1/2-1/u} & \text{if } 1 \leq u \leq 2 \leq v \leq \infty, \\ n^{2/r} & \text{if } 2 \leq u, v \leq \infty. \end{cases}$$

Using (4.1), these estimates can be improved for those u, v for which $\lambda_\ell(\Pi_{r,2}, u, v) = 0$. $\square$

Our results for $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v)$ can be summarized in the following picture:



The contents of this article will be part of the second named author's thesis written at the Carl von Ossietzky University of Oldenburg under the supervision of the first named author.

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